55^{th} Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

29th/30th May 2024

Problem 1. *Determine the smallest constant C such that the inequality*

$$(X+Y)^2(X^2+Y^2+C)+(1-XY)^2 \geq 0$$

holds for all real numbers X and Y.

For which values of X and Y does equality hold for this smallest constant C?

(Walther Janous)

Answer. The smallest constant is $C=-1$. Equality holds for $X=Y=1/\sqrt{3}$ or $X=Y=-1/\sqrt{3}$.

Solution. We first investigate the case $X=Y$. It is easily seen that the inequality becomes equivalent to

$$(3X^2-1)^2+4(C+1)X^2 \geq 0$$

which implies $C \geq -1$ by setting $X^2=\frac{1}{3}$.

It remains to prove that the inequality is true for all X and Y for $C=-1$.

Since we had the term $(3X^2-1)^2$ in the above case, we compare the term $(X^2+XY+Y^2-1)^2$ with the terms in the given inequality and get the equivalent inequality

$$(X^2+XY+Y^2-1)^2+(X-Y)^2 \geq 0.$$

This is obviously true and gives the conditions $X=Y$ and $3X^2-1=0$ for equality which are the two cases $X=Y=\frac{1}{\sqrt{3}}$ and $X=Y=-\frac{1}{\sqrt{3}}$.

(Theresia Eisenkölbl) □

Problem 2. *Let ABC be an acute triangle with $AB > AC$. Let D, E and F denote the feet of its altitudes on BC, AC and AB, respectively. Let S denote the intersection of lines EF and BC.*

Prove that the circumcircles k_1 and k_2 of the two triangles AEF and DES touch in E.

(Karl Czakler)

Solution. Let t_1 be the tangent line to k_1 in point E and let t_2 be the tangent line to k_2 in point E. The tangent-secant theorem applied to circle k_1 gives

$$\angle(EF,t_1)=\angle FAE=\alpha$$

with the usual notation for the angles in triangle ABC.

The tangent-secant theorem applied to circle k_2 gives

$$\angle(EF,t_2)=\angle SDE=\angle CDE=\alpha,$$

where the last equality comes from the fact that $ABDE$ is a cyclic quadrilateral since all four vertices lie on the Thales circle with diameter AB.

Therefore, t_1 and t_2 are parallel and they both contain the point E. So, the two tangents are identical which implies that the circles touch in E.

(Theresia Eisenkölbl) □

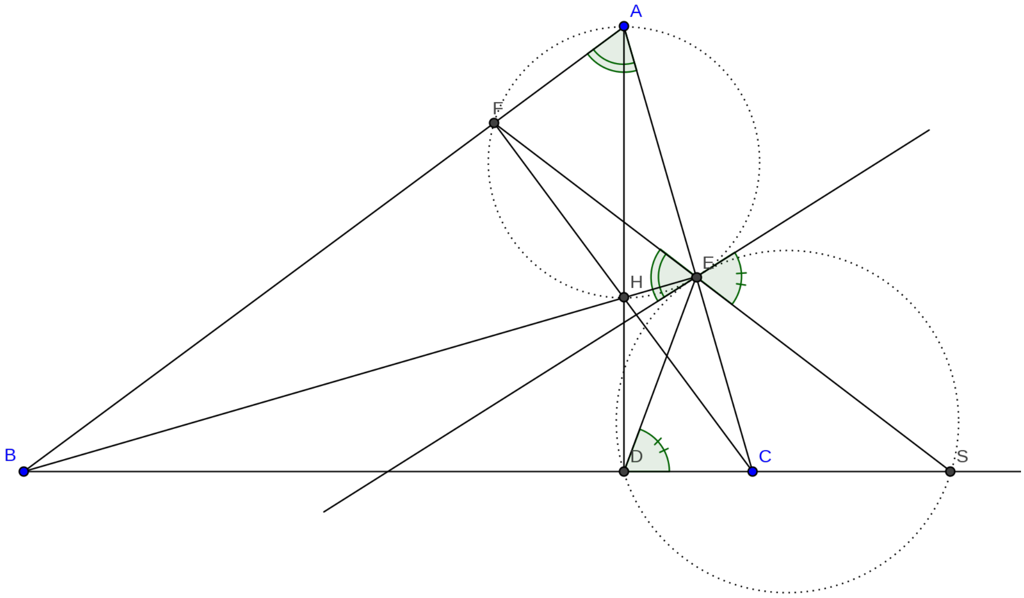

Figure 1: Problem 2

Problem 3. *Initially, the numbers* 1, 2, ..., 2024 *are written on a blackboard. Trixi and Nana play a game, taking alternate turns. Trixi plays first.*

The player whose turn it is chooses two numbers a *and* b*, erases both, and writes their (possibly negative) difference* $a - b$ *on the blackboard. This is repeated until only one number remains on the blackboard after* 2023 *moves. Trixi wins if this number is divisible by* 3*, otherwise Nana wins.*

Which of the two has a winning strategy?

(Birgit Vera Schmidt)

Solution. We will prove that Nana has a winning strategy.

The only relevant property of all numbers in the game is their residue modulo 3. Therefore, we will call all numbers 0, 1 or 2 according to their residue, and we will also call 1s and 2s non-zeros.

We observe that each move either does not change the number of non-zeros (if one or two zeros are involved in the move) or decreases the number of non-zeros by 1 or 2 (if no zero is involved in the move).

Nana can play arbitratrily for a long time while the number of non-zeros decreases, until that number reaches 1, 2, 3 or 4 at the start of her move. This has to happen because it is not possible to go from 5 or more non-zeros to 0 non-zeros in two moves, and Trixi certainly cannot win as long as there are non-zeros on the blackboard.

If the number of non-zeros is 4, then Nana will avoid decreasing the number or non-zeros by using one or two zeros to force Trixi to decrease the number to 2 or 3. This has to happen because Trixi always starts a move with an even quantity of numbers, so she is the first one without zeros as long as there are 4 non-zeros.

If the number of non-zeros is 3, then two of them have the same value. Nana chooses these two and replaces them with zero. This leaves one non-zero which can change between 1 and 2, but never be removed until the end. So Nana wins.

If the number of non-zeros is 2, and they are distinct, then Nana replaces them with their difference 1 which again can never become zero.

If the number of non-zeros is 2 and they have the same value, then Nana will use one of them and a 0 to convert them to $(1, 2)$. This is possible because Nana always starts her move with an odd quantity of numbers, so she certainly has an available 0. If Trixi uses $(1, 2)$, she will lose since the last non-zero cannot be converted to zero. She also cannot use two zeros, because then Nana is in the previous case and wins. So Trixi has to convert one of them with an additional 0 to present Nana with two equal non-zeros. However, Nana can repeat her move until Trixi has not zeros left to do so. So Trixi will eventually be forced to use $(1, 2)$ and loses.

If there is just one non-zero left, Nana can play arbitrarily because this single non-zero will remain until the end of the game.

(Theresia Eisenkölbl) □

Problem 4. *Let ABC be an obtuse triangle with orthocenter H and centroid S. Let D, E and F be the midpoints of segments BC, AC, AB, respectively.*

Show that the circumcircle of triangle ABC, the circumcircle of triangle DEF and the circle with diameter HS have two distinct points in common.

(Josef Greilhuber)

Solution. Let M and O denote the circumcenters of the triangles DEF and ABC, respectively. We will use the well-known facts that the points H, M, S and O lie on the Euler line of triangle ABC in this order, and that $HM : MS : SO = 3 : 1 : 2$.

Since S is in the interior of ABC, it is inside the circumcircle of ABC. However, since ABC is obtuse, the orthoscenter H is outside the circumcircle. This implies that the circle with diameter HS intersects the circumcircle of ABC in two points.

Let X be one of these intersection points. We will prove that $MX : OX = 1 : 2$. Let N be the midpoint of HS. Using Thales' theorem, we get that HN, XN and SN have the same length, and $HN : NM : MS : SO = 2 : 1 : 1 : 2$. This implies that $MN : XN = XN : ON$. Therefore, the triangles XNM and ONX are similar with ratio $1 : 2 = MN : XN$. Therefore, we have $MX : OX = 1 : 2$ as desired.

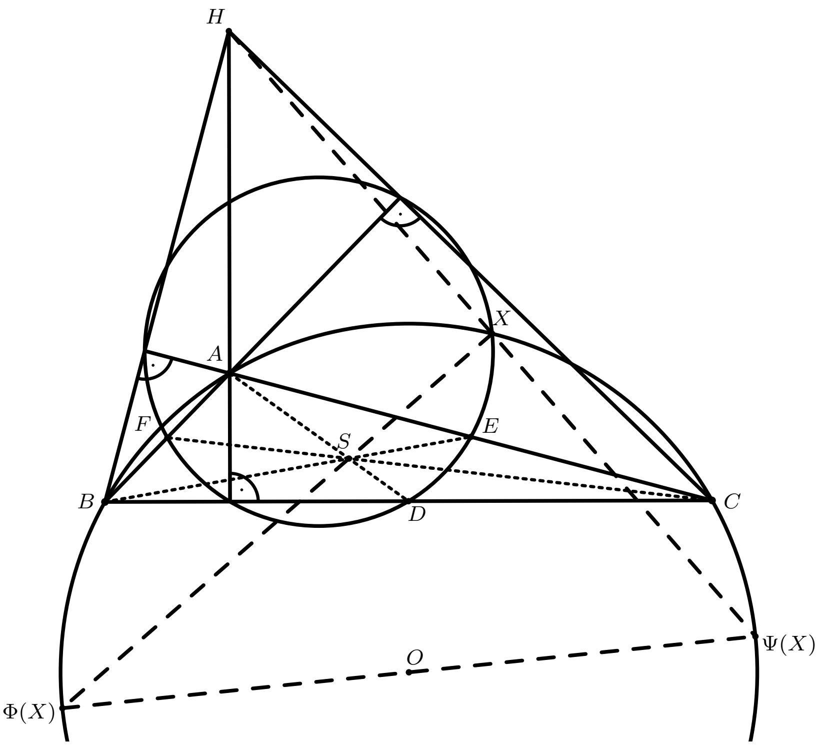

Figure 2: Figure for Problem 4

(Josef Greilhuber) □

Problem 5. *Let n be a positive integer and let z_1, z_2, ..., z_n be positive integers such that for $j = 1, 2, \ldots, n$ the inequalities*

$$z_j \leq j$$

hold and $z_1 + \ldots + z_n$ is even.

Prove that the number 0 occurs among the values of

$$z_1 \pm z_2 \pm \ldots \pm z_n,$$

where + or − can be chosen independently for each operation.

(Walther Janous)

Solution. We carry out the proof with complete induction.

- $n = 1$. Here $z_1 = 1$ and the sum cannot be even, so there is nothing to prove.
- $n = 2$. Here $z_1 \leq 1$, $z_2 \leq 2$ and the condition that $z_1 + z_2$ is even lead to $z_1 = z_2 = 1$ together with $z_1 - z_2 = 0$.
- $n = 3$. Here $z_1 \leq 1$, $z_2 \leq 2$, $z_3 \leq 3$ and the condition that $z_1 + z_2 + z_3$ is even result in the three possibilities $(1, 1, 2)$ and $(1, 2, 1)$ or $(1, 2, 3)$ with $1 + 1 - 2 = 0$, $1 - 2 + 1 = 0$ and $1 + 2 - 3 = 0$.
- Suppose that the statement holds up to n and now draw the conclusion from n to $n + 1$. We distinguish between two cases.

  (a) $z_{n+1} = z_n$: In this case, $z_1 + \ldots + z_{n-1}$ is even. Therefore, 0 can be represented in the form $z_1 \pm \ldots \pm z_{n-1}$ and thus also as $z_1 \pm \ldots \pm z_{n-1} + z_n - z_{n+1}$.

  (b) $z_{n+1} \neq z_n$: With $z_1 + \ldots + z_{n-1} + z_n + z_{n+1}$ the sum $z_1 + \ldots + z_{n-1} + |z_{n+1} - z_n|$ is also even. Furthermore, $1 \leq |z_{n+1} - z_n| \leq |(n + 1) - 1| = n$ and we so can apply the induction assumption for the n numbers $z_1, \ldots, z_{n-1}, |z_{n+1} - z_n|$. Consequently, 0 can be represented as $z_1 \pm \ldots \pm z_{n-1} \pm |z_{n+1} - z_n|$. Because $|z_{n+1} - z_n| = \pm(z_{n+1} - z_m)$, we end up with a representation of 0 in the form $z_1 \pm \ldots \pm z_{n-1} \pm z_n \pm z_{n+1}|$, which completes the induction step.

(Walther Janous) □

Problem 6. *For each prime number p, determine the number of residue classes modulo p which can be represented as $a^2 + b^2$ modulo p, where a and b are arbitrary integers.*

(Daniel Holmes)

Answer. All p residue classes.

Solution. With $a^2 + 0^2$ we first obtain all quadratic residue classes.

Since not all residue classes are quadratic residues, there is a quadratic residue class a^2 that is followed by a quadratic non-residue class, so that $n = a^2 + 1$ is not a quadratic residue and therefore of course $n \not\equiv 0 \pmod{p}$.

However, since the product of two quadratic non-residue classes is a quadratic residue class, it follows for each quadratic non-residue class m that $m = nmn/n^2 = (a^2 + 1)c^2/n^2 \equiv (acn^{-1})^2 + (cn^{-1})^2 \pmod{p}$ and therefore all quadratic residue classes can also be represented as the sum of two squares.

(Theresia Eisenkölbl) □