46^{th} Austrian Mathematical Olympiad

Regional Competition (Qualifying Round) – Solutions

March 26, 2015

Problem 1. *Determine all triples (a, b, c) of positive integers satisfying the conditions*

$$\begin{aligned} \gcd(a, 20) &= b, && \text{(I)}\\ \gcd(b, 15) &= c \qquad \textit{and} && \text{(II)}\\ \gcd(a, c) &= 5. && \text{(III)} \end{aligned}$$

(Richard Henner)

Solution. We use equations (I) und (II) in order to eliminate b and c as follows:

$$\gcd(a, \gcd(\gcd(a, 20), 15)) = 5 \iff \gcd(a, a, 20, 15) = 5 \iff \gcd(a, 5) = 5 \iff 5 \mid a.$$

Furthermore we determine b and c from (I) and (II): (I) yields $b \in \{5, 10, 20\}$. More specifically we have $b = 5$ for a being odd, $b = 10$ for $a \equiv 2 \bmod 4$ and $b = 20$ for $a \equiv 0 \bmod 4$. In all three cases $c = 5$ follows from (II).

In total the solutions form the set $\{(20t, 20, 5), (20t - 10, 10, 5), (10t - 5, 5, 5) \mid t \text{ is a positive integer}\}$.

(Walther Janous, Gerhard Kirchner) □

Problem 2. *Let x, y and z be positive real numbers with $x + y + z = 3$.*
Prove that at least one of the three numbers

$$x(x + y - z), \qquad y(y + z - x) \qquad \textit{or} \qquad z(z + x - y)$$

is less or equal 1.

(Karl Czakler)

Solution. Since the three expressions are cyclic, we may w. l. o. g. assume that $x \geq y, z$. Consequently we have $x \geq \frac{x+y+z}{3} = 1$. We now show that $a := y(y + z - x) = y(3 - 2x)$ satisfies $a \leq 1$.

- *Case a) :* For $\frac{3}{2} \leq x < 3$ clearly $a \leq 0 < 1$.
- *Case b) :* For $1 \leq x < \frac{3}{2}$ the factor $3 - 2x$ is positive. Therefore $a \leq x(3 - 2x)$. Hence it suffices to prove $x(3 - 2x) \leq 1$, which is equivalent to $2x^2 - 3x + 1 \geq 0$, i. e. $(2x - 1)(x - 1) \geq 0$.

This completes the proof.

(Walther Janous) □

Problem 3. *Let $n \geq 3$ be a fixed integer. The numbers $1, 2, 3, \ldots, n$ are written on a board. In every move one chooses two numbers and replaces them by their arithmetic mean. This is done until only a single number remains on the board.*

Determine the least integer that can be reached at the end by an appropriate sequence of moves.

(Theresia Eisenkölbl)

Solution. The answer is 2 for every n. Surely we cannot reach an integer less than 2, since 1 appears only once and produces an arithmetic mean greater than 1, as soon as it is used.

On the other hand, we can prove by induction on k that the number $a+1$ can be reached from the numbers a, $a+1$, …, $a+k$ by a sequence of permitted moves.

For $k=2$ one replaces a and $a+2$ by $a+1$ and afterwards $a+1$ and $a+1$ by a single $a+1$.

For the induction step $k \to k+1$ one replaces $a+1$, …, $a+k+1$ by $a+2$ and afterwards a and $a+2$ by $a+1$.

In particular with $a=1$ and $k=n-1$ one achieves the desired result.

(Theresia Eisenkölbl) □

Problem 4. *Let ABC be an isosceles triangle with $AC = BC$ and $\angle ACB < 60°$. We denote the incenter and circumcenter by I and O, respectively. The circumcircle of triangle BIO intersects the leg BC also at point $D \neq B$.*

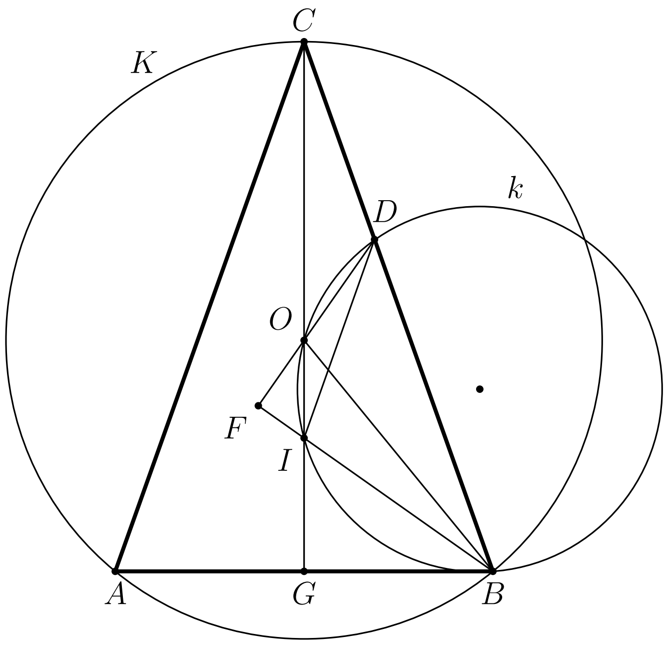
(a) Prove that the lines AC and DI are parallel.

(b) Prove that the lines OD and IB are mutually perpendicular.

(Walther Janous)

Solution. Note that the condition $\angle ACB < 60°$ guarentees that O lies between I and C.

a) We denote the angles of triangle ABC by $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$. Let K and k be the circumcircles of ABC and BIO, respectively. The inscribed angle theorem for circle K yields: $\angle BOC = 2\alpha$. Therefore we have $\angle IOB = 180° - 2\alpha$ and because of $\alpha = \beta$ we obtain $\angle IOB = \gamma$. Furthermore the inscribed angle theorem for circle k gives $\angle IDB = \gamma$, whence finally $ID \parallel AC$.



b) We denote the point of intersection of lines OD by F and IB and the midpoint of AB by G. Since $IODB$ is cyclic, we have $\angle IOD = 180° - \beta/2$, that is $\angle DOC = \beta/2$ or equivalently $\angle FOI = \beta/2$. Furthermore $\angle GIB = 90° - \beta/2$ implies $\angle OIF = 90° - \beta/2$. Therefore $\angle IFO = 90°$.

(Richard Henner) □