50^{th} Austrian Mathematical Olympiad

Regional Competition—Solutions

4th April 2019

Problem 1. *Let x and y be real numbers satisfying $(x+1)(y+2) = 8$.*

Show that

$$(xy - 10)^2 \geq 64.$$

Furthermore, determine all pairs (x, y) of real numbers for which equality holds.

(Karl Czakler)

Solution. The inequality $(2x - y)^2 \geq 0$ (with equality if and only if $y = 2x$) is equivalent to

$$(2x + y)^2 \geq 8xy.$$

The constraint $(x+1)(y+2) = 8$ gives $2x + y = 6 - xy$. Substituting this into the inequality above yields

$$(6 - xy)^2 \geq 8xy,$$

which is equivalent to

$$(xy - 10)^2 \geq 64.$$

As we noted already, equality holds for $y = 2x$. In this case, the constraint becomes $(x+1)(2x+2) = 8$ which yields $x = 1$ or $x = -3$ and finally the two pairs $(x, y) = (1, 2)$ and $(x, y) = (-3, -6)$. We easily verify that equality actually holds in both cases.

(Karl Czakler) □

Problem 2. *Let $ABCDE$ be a convex pentagon having a circumcircle and satisfying $AB = BD$. The point P is the intersection of the diagonals AC and BE. The lines BC and DE intersect in point Q.*

Show that the line PQ is parallel to the diagonal AD.

(Gottfried Perz)

Solution. We denote the circumcircle of the pentagon $ABCDE$ by k, see Figure 1. By assumption, the triangle ABD is isosceles, which implies that the tangent t_B to k in B is parallel to AD.

We apply Pascal's theorem to the inscribed hexagon $BEDACB$: The intersection point of the opposite sides BE and AC is P, the intersection point of the opposite sides ED and CB is Q, and the intersection point of the parallel opposite sides BB (i.e., t_B) and DA is the point at infinity corresponding to direction AD. Therefore, PQ is parallel to AD.

(Stefan Leopoldseder) □

Problem 3. *Let $n \geq 2$ be an integer.*

We draw an $n \times n$ grid on a board and label each box with either the number -1 or the number 1. Then we calculate the sum of each of the n rows and the sum of each of the n columns and determine the sum S of these $2n$ sums.

(a) Show that there does not exist a labelling of the grid with $S = 0$ if n is odd.

(b) Show that there exist at least six different labellings with $S = 0$ if n is even.

(Walther Janous)

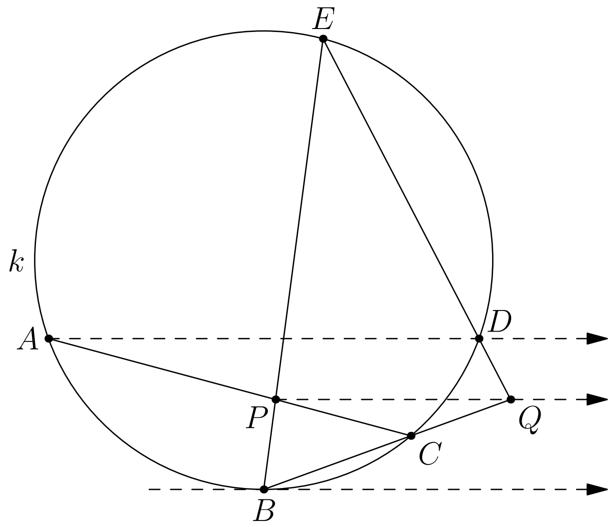

Figure 1: Problem 2

Solution. As each number of the grid appears exactly once in the sum of all columns of the grid and the same holds for the sum of all rows, we get that S is twice the sum of all labels of the boxes of the $n \times n$ grid. Therefore, $S = 0$ holds if and only if the sum of all labels of the boxes vanishes, or equivalently, if the number of labels $+1$ equals the number of labels -1. We call such a labelling *admissible.*

(a) If n is odd, the sum of all labels is also odd, because it is a sum of an odd number of odd labels. Thus there cannot be an admissible labelling in this case.

(b) If n is even, we write $n = 2k$ for some integer k. The admissible labellings can be constructed as follows: Choose exactly half of the $n^2 = 4k^2$ boxes arbitrarily and label each of them with $+1$. The remaining boxes are labelled with -1.

Thus there are exactly $a_k := \binom{4k^2}{2k^2}$ admissible labellings of a $2k \times 2k$ grid.

We have $a_1 = \binom{4}{2} = 6$ and it is easily seen that a_k is increasing in k: if $1 \leq k' < k$, each admissible labelling of any $2k' \times 2k'$ subgrid can be extended to an admissible labelling of the $2k \times 2k$ grid by choosing half of the extra $4k^2 - 4k'^2$ boxes and labelling each of them with $+1$ and the remaining boxes with -1. Therefore, $a_k \geq 6$ for all k.

(Walther Janous) □

Problem 4. *Determine all non-negative integers n smaller than 128^{97} which have exactly 2019 positive divisors.*

(Richard Henner)

Answer. There are 4 solutions: $n = 2^{672} \cdot 3^2$ or $n = 2^{672} \cdot 5^2$ or $n = 2^{672} \cdot 7^2$ or $n = 2^{672} \cdot 11^2$.

Solution. Numbers with exactly 2019 positive divisors are either of the form p^{2018} or $p^{672} \cdot q^2$ for distinct prime numbers p and q. The number 128^{97} can be written as

$$128^{97} = (2^7)^{97} = 2^{679}.$$

As p is at least 2, the number p^{2018} is greater than 2^{679} and therefore, the case $n = p^{2018}$ is impossible. Thus we have $n = p^{672} \cdot q^2$ with $p^{672} \cdot q^2 < 2^{679}$. Hence $p = 2$ and as $q^2 < 2^7 = 128$, q is one of the primes 3, 5, 7 or 11.

(Richard Henner) □