54^{th} Austrian Mathematical Olympiad

Junior Regional Competition—Solutions

13th June 2023

Problem 1. *Let x, y, z be nonzero real numbers with*

$$\frac{x+y}{z} = \frac{y+z}{x} = \frac{z+x}{y}.$$

Determine all possible values of

$$\frac{(x+y)(y+z)(z+x)}{xyz}.$$

(Walther Janous)

Answer. The only possible values are 8 and -1.

Solution. We add 1 to the equations and obtain

$$\frac{x+y+z}{z} = \frac{x+y+z}{x} = \frac{x+y+z}{y}.$$

For $x+y+z=0$, this is clearly true, and the expression in the problem statement becomes

$$\frac{(-z)(-x)(-y)}{xyz} = -1.$$

For $x+y+z \neq 0$, we get

$$\frac{1}{z} = \frac{1}{x} = \frac{1}{y}$$

and therefore $x = y = z$.

In this case, our expression becomes 8.

The values -1 and 8 are attained because any triple with $x+y+z=0$ resp. $x=y=z$ that does not contain a zero works.

(Theresia Eisenkölbl) □

Problem 2. *Let $ABCDEF$ be a regular hexagon with sidelength s. The points P and Q are on the diagonals BD and DF, respectively, such that $BP = DQ = s$.*

Prove that the three points C, P and Q are on a line.

(Walther Janous)

Solution. Our strategy is to compute the angles $\angle DCQ$ and $\angle DCP$ to check that they are equal.

The interior angles of a regular hexagon equal 120°. The triangle DEF is isosceles and therefore, we get $\angle DFE = \angle EDF = 30°$. This implies $\angle QDC = 90°$, and since the triangle QDC is also isosceles, we also get

$$\angle DCQ = 45°.$$

The triangle CBP is isosceles and analogously to the above, we get $\angle CBP = \angle CBD = 30°$. Therefore, we obtain

$$\angle PCB = (180° - 30°) : 2 = 75° \quad \text{and, finally,} \quad \angle DCP = 120° - 75° = 45°.$$

So $\angle DCQ = \angle DCP$ which implies that C, P and Q lie on a line.

(Walther Janous) □

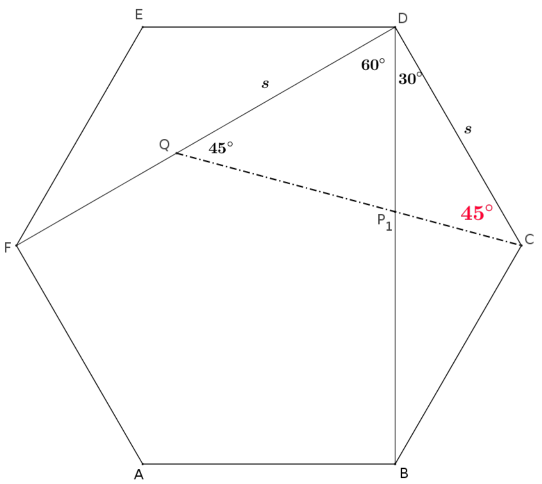
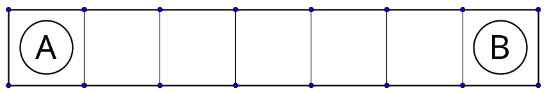


Figure 1: Problem 2



Problem 3. *Alice and Bob play a game on a strip of $n \geq 3$ squares with two game pieces. At the beginning, Alice's piece is on the first square while Bob's piece is on the last square. The figure shows the starting position for a strip of $n = 7$ squares.*

The players alternate. In each move, they advance their own game piece by one or two squares in the direction of the opponent's piece. The piece has to land on an empty square without jumping over the opponent's piece. Alice makes the first move with her own piece. If a player cannot move, they lose.

For which n can Bob ensure a win no matter how Alice plays?

For which n can Alice ensure a win no matter how Bob plays?

(Karl Czakler)

Answer. Bob wins for $n = 3k + 2$ with $k \in \mathbb{Z}_{\geq 1}$, Alice wins for all other $n \geq 3$.

Solution. It is easily checked that Alice wins for $3, 4, 6$ and 7 squares while Bob wins for 5 or 8 squares. We conjecture that Bob wins for all n of the form $3k + 2$ and prove it by induction.

We include the case $k = 0$ which is obvious because Alice loses immediately.

Now, we assume that Bob can assure a win for $3k + 2$ squares for a certain natural number k. Now, we want to prove that he can ensure a win for $3k + 5$.

It is enough that Bob makes exactly the opposite move of Alice after her first move: If she moves by 1, he moves 2. If she moves by 2, he moves 1. This ensures that the distance between the two game pieces is reduced by 3 and the game continues as if it were a new game with $3k + 2$ squares where we already know that Bob can ensure a win.

Therefore, we have proved that Bob can win for all $n = 3k + 2$.

Now, it remains to show that Alice can win for all n of the form $3k$ and $3k + 1$.

In the case of $3k$ squares, she starts the game by moving 1 such that the remaining game is played on $3k - 1 = 3(k - 1) + 2$ squares with Bob making the first move. So we already know that Alice as the second player can ensure a win.

In the case of $3k + 1$, Alice starts with 2 which again reduces the game to a game with $3(k - 1) + 2$ squares with Bob making the first move.

We can conclude that Bob can ensure a win for all $n = 3k + 2$, and Alice can ensure a win for all other n.

(Theresia Eisenkölbl) □

Problem 4. *Determine all triples* (a, b, c) *of positive integers such that*

$$a! + b! = 2^{c!}.$$

(Walther Janous)

Answer. The only solutions are $(1, 1, 1)$ and $(2, 2, 2)$.

Solution. We can assume without loss of generality that $a \leq b$.

- For $a = b = 1$, we get $c = 1$, which gives the solution $(1, 1, 1)$.
- For $a = 1$ and $b > 1$, the left-hand side is bigger than 1 and odd, therefore, it cannot be a power of 2 and we do not get a solution in this case.
- For $a = b = 2$, we get $c = 2$, therefore $(2, 2, 2)$ is a solution.
- For $a = 2$ and $b = 3$, we get $2! + 3! = 8 = 2^3$. But there is no c with $c! = 3$. Therefore, there is no solution in this case.
- For $a = 2$ and $b \geq 4$, we get $2! + b! \geq 2! + 4! = 26$. Therefore, we have $c! > 4$. This implies that the left-hand side is congruent to 2 modulo 4, while the right-hand side is congruent to 0 modulo 4. Therefore, there is no solution in this case.
- For $a \geq 3$ and $b \geq 3$, the left-hand side is divisible by 3 while the power of 2 on the right-hand side is not. Therefore, there is no solution in this case.

(Reinhard Razen) □